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Topology

Theorem- Let X be a topological space. Then the class of closed subsets of X possesses the following properties.

(i) ϕ and X are closed sets.

(ii) The intersection of any number of closed sets is closed.

(iii) The union of any two sets is closed.

Proof:- (i) Since $\phi, X \in T$ and ϕ and X are both open as well as closed

(ii) Let $\{A_i\}$ be the collection of members of T then $\cup_i A_i \in T$. i.e. $\cup_i A_i$ is open (union of open sets is open)

Taking complement and using De Morgan's Law we get

$(\cup_i A_i)^c = \cap_i A_i^c$ is closed.

$\Rightarrow A_i$ is open $\forall i$.

$\Rightarrow A_i^c$ is closed $\forall i$.

Hence intersection of any number of closed sets is closed

(iii) Let $A, B \in T \Rightarrow A \cap B \in T$. And since $A, B \in T$ then both A and B are open. Also since intersection of open sets is also open, so $A \cap B$ is open.

$\Rightarrow A^c, B^c$ and $(A\cap B)^c$ are closed

$(A \cap B)^c = A^c \cup B^c$

$\Rightarrow$ Union of any two closed sets is closed in (X, T)

Hence the class of closed subsets of X possesses all the three properties describe in statement

Theorem:- A subset A of a topological space X is closed iff A contains each of its limit point.

Proof:- Assume that A is closed, then we are to show that $A' \subset A$.

Let p be a limit point of A such that $p \notin A$ then $p \in A^c$. But A^c is open since A is closed.

Hence $p \notin A'$ for A^c is open set, such that

$$p \in A^c \quad \text{and} \quad A^c \cap A = \phi$$

$$\Rightarrow (A^c - \{p\}) \cap A = \phi$$

which is a contradiction to the fact that $p \notin A$.

Thus $A' \subset A$ if A is closed.

Conversely

Now assume that $A' \subset A$, then we are to show that A is closed

For this, we show that A^c is open.

Let $p \in A^c$ then $p \notin A$, so $\exists$ an open set G such that $p \in G$ and

$$(G - \{p\}) \cap A = \phi$$

But $p \notin A$, hence

$$G \cap A = \phi$$

So, $G \subset A^c$.

Thus p is an interior point of A^c and so A^c is open.

$\Rightarrow$ A is closed

; proved.